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AUTHOR	Barksdale, James B., Jr.
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ABSTRACT

The author describes the concept of Randolph diagrams (R-diagrams) which is a scheme for simplifying set-theoretic and/or logical expressions. Several applications of R-diagrams are presented. [Not available in hardcopy due to marginal legibility of original document.] (FL)

SETS AND RANDOLPH DIAGRAMS

by

James B. Barksdale, Jr.

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Western Kentucky University

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Abstract

In this exposition the author discusses the notion and application of Randolph diagrams (R-diagrams). The notions discussed here first became known to the author when Professor Randolph delivered a lecture during the summer of 1965 at the University of Arkansas.

The concept of R-diagrams provides a rather clever and simple scheme for simplifying set-theoretic and/or logical expressions. Other applications in this general regard are easily conceived. The reader will find some of these applications illustrated in this presentation.

Sets and Randolph Diagrams

James B. Barksdale, Jr.
Western Kentucky University

Venn-Euler diagrams (VE-diagrams) provide an excellent means of illustrating the fundamental set-theoretic notions such as: set inclusion, complementation, unions and intersections. The value of the Venn-Euler diagram is, of course, that the student can see a "picture" of the situation involving the two or three sets, and can better determine how the various parts of these sets are related to each other. VE-diagrams may also be used to perform simple set-theoretic operations involving one or two sets. If, however, more than two or three sets are involved, VE-diagrams become very difficult to write down, and it becomes very difficult to distinguish the various parts of the sets involved. To say the least, it becomes extremely tedious to perform even the simplest set-theoretic operations when more than two sets are involved. In order to obtain a method which offers the advantages of the VE-diagrams, and at the same time offers greater facility in computation, we must examine the fundamental aspects of the VE-diagrams, and then be sure to incorporate these features into the improved method.

We begin with a close examination of VE-diagrams.

Let U denote the universal set whose boundary is taken to be rectangular. Note that when a set A is introduced, the circular boundary for A divides U into two disjoint (separate) parts (see Figures 1 and 2); namely, the part interior to A (i.e., the set A), and the part exterior to A (i.e., the set A').

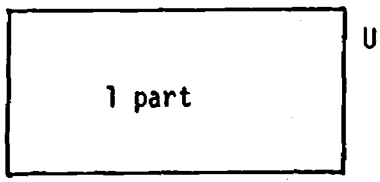


Fig. 1
Universal Set

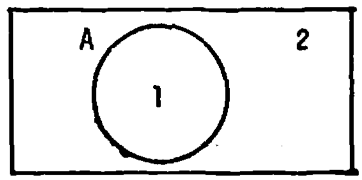


Fig. 2
The boundary A splits U into two disjoint parts "1" and "2".

When a second set B is introduced, the boundary B divides each of the two parts in Fig. 2 into two parts (see Fig. 3). Namely, we speak of that portion of part-1 which is in B, and that portion of part-1 which is not in B. Similarly, we may speak of that portion of part-2 which is in B, and that portion of part-2 which is not in B. Altogether, we have four disjoint parts.

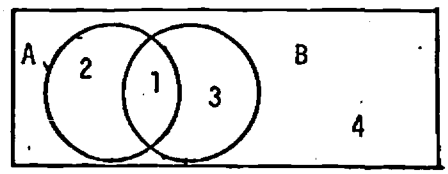


Fig. 3

Two boundaries divide U into four disjoint parts.

In a similar fashion, if a third set C were introduced, then each of the four disjoint parts in Fig. 3 would be divided into two portions (see Fig. 4).

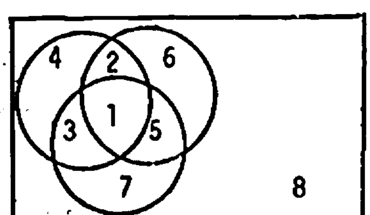


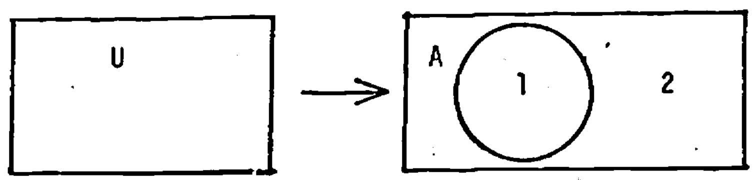
Fig. 4

Three boundaries divide U into eight disjoint parts.

We see, then, that the introduction of a single boundary divides each of the existing disjoint parts into two portions. If n-boundaries were introduced, then there would be 2^n disjoint portions. These last two statements constitute the fundamental features of VE-diagrams.

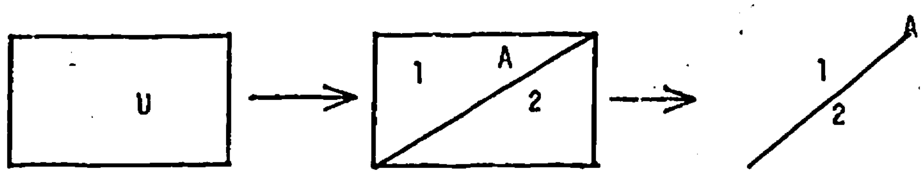
We now proceed to design a method which will incorporate these features, but at the same time remain simpler and more functional. Consider the following sequences of diagrams.

S_1:



Frame 1 Frame 2

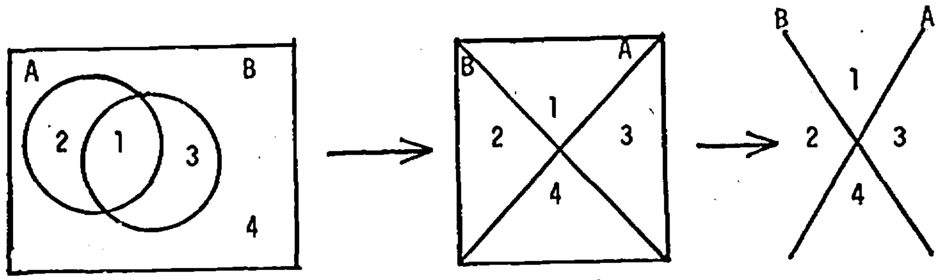
S_2:



Frame 1 Frame 2 Frame 3

The first sequence S_1 simply indicates that if we start with the universal set and introduce a set A, then U will be divided into two parts. The second sequence S_2 indicates that if we start with the universal set, then we may divide U into two parts with the introduction of a "straight line" boundary for A. That portion <u>above</u> the boundary is in the set A, and that portion <u>below</u> the boundary is <u>not</u> in A (i.e., is in A'). The third frame of the sequence S_2 simply indicates that we may leave off the rectangular boundary for U. Now consider the sequence S_3, below.

S_3:

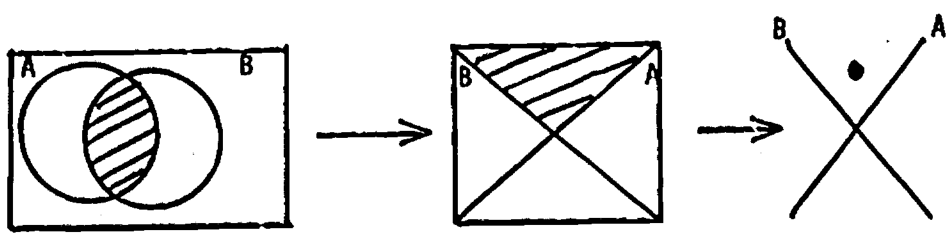


Frame 1 Frame 2 Frame 3

The first frame of S_3 simply denotes that U has been split into four disjoint parts by the introduction of boundaries A and B. Frame 2 of S_3 indicates that the division of U into four disjoint parts may be accomplished by introducing a second diagonal "straight line" boundary B (i.e., compare Frame 2 of S_2 with Frame 2 of S_3). Regarding this second "straight line" boundary B, the portion <u>above</u> B is considered to be in the set B, and that portion <u>below</u> B is not in the set B (i.e., is in B'). Frame 3 of S_3 simply indicates that the rectangular boundary of U is really not necessary.

We now consider one more sequence of frames S_4, below.

S_4:



Frame 1 Frame 2 Frame 3

The first frame of S_4 is a VE-diagram for $A \cap B$, since that portion which is in A <u>and</u> at the same time in B has been shaded. The second frame of S_4 shows the corresponding diagram using "straight line" boundaries. Note that $A \cap B$, here, is that portion which is <u>above</u> A and at the same time <u>above</u> B. Frame 3 of S_4 is obtained by first deleting the rectangular boundary for U, and then by using a "dot" in the place of "shading". Hence,

(1) $A \cap B =$ 

The diagram in (1) will be referred to as the R-diagram (Randolph diagram) for the set $A \cap B$.

<u>Remark</u>. The reader is reminded here that any two sets A and B always intersect. This intersection may, of course, be the null set. Hence, when referring to Frame 1 of the sequence S_4, the reader should note that the shaded area indicates only the intersection $A \cap B$, and does <u>not</u> signify that there are actually any elements in this intersection. Similarly, the presence of the dot in the third frame of S_4 only denotes the intersection $A \cap B$. The presence of this dot does not, however, indicate that $A \cap B$ is non-empty.

Other R diagrams we may consider are given below.

(2) $A' \cap B$ =  The dot is placed below A and at the same time above B.

(3) $A \cap B'$ =  The dot is placed below B and at the same time above A.

(4) $A' \cap B'$ =  The dot is placed below A and at the same time below B.

In order to represent the set A, all portions above the boundary A must receive a dot. Hence,

(5) A = 

In order to represent the set B', all portions below the boundary B must receive a dot. Thus,

(6) B' = 

In order to represent the set $A \cup B$, we must first dot in all portions of A, and then continue dotting until all portions of B are also dotted in. That is, to denote $A \cup B$, we must dot in all of A together with all of B. Hence,

(7) $A \cup B =$ 

We now list some operational rules for R-diagrams.

<u>Rule I.</u> To <u>complement</u> a R-diagram, reverse the roles of "dots" and "blanks" in the diagram.

For example,

(8) $(A \cup B)' =$ ()' $=$  $= A' \cap B'$.

<u>Rule II.</u> To <u>union</u> two or more R-diagrams, write a R-diagram which has a dot occurring in exactly those places where <u>at least one</u> of the diagrams in the union has a dot.

For example,

(9) $A' \cup B \cup (A \cap B) =$  $\cup$  $\cup$  $=$ 

<u>Rule III.</u> To <u>intersect</u> two or more R-diagrams, write a R-diagram which has dots in <u>only</u> those places where

<u>all</u> diagrams have a dot occurring.

For example,

(10) $(A \cap B) \cap (A' \cup B')$ =  $\cap$  =  = $\emptyset$.

<u>Rule IV.</u> Let R(S) denote the R-diagram for the set S. Then, $A \subseteq B$ exactly when "For each dot in R(A), there is a corresponding dot in R(B)".

<u>Rule V.</u> Two sets are <u>equal</u> whenever they have the same R-diagram.

The above rules follow directly from the definitions of union, intersection, set inclusion, etc.. We now consider several examples illustrating the uses of these rules.

EXAMPLE 1. Write the R-diagram for: (a) $A' \cup B$, and (b) $A' \cup B'$.

SOLUTION. (a) To write the R-diagram for $A' \cup B$, we appeal to Rule II and dot in all portions <u>below A</u>, and then continue until all portions <u>above B</u> have also been dotted in. Hence,

(11) $A' \cup B$ = 

(b) To write the R-diagram for $A' \cup B'$, we use Rule II again. Hence,

(12) $A' \cup B' =$ 

EXAMPLE 2. Write the set indicated by the following R-diagrams.

(a)    (b) 

SOLUTION. (a) We note that a dot has been placed in the portion representing $A \cap B$, and a dot has been placed so that it is below A and below B, which is the portion denoting $A' \cap B'$. Hence,

(13)  $= (A \cap B) \cup (A' \cap B')$.

Note that we could also write,

(14)  $=$  $\cup$  $= (A \cap B) \cup (A' \cap B')$.

(b) Here, we see that all portions above A have dots. Also, all portions below B have dots. Hence, we have all of A together with all of B'. Thus,

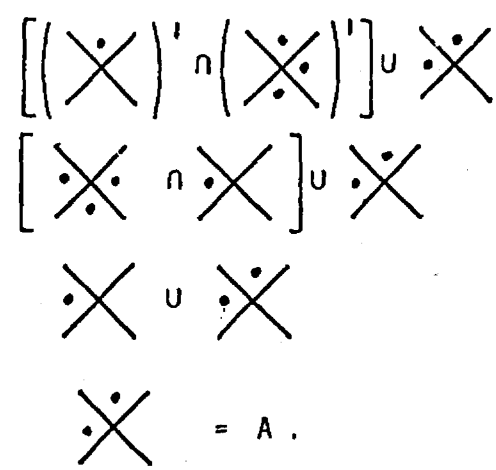
(15)  $= A \cup B'$.

Note that we could write,

(16)  $= A \cup B'$.

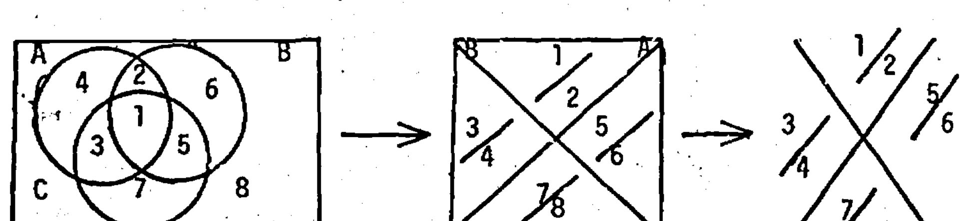
EXAMPLE 3. Simplify the expression: $[(A \cap B)' \cap (A' \cup B)'] \cup A$.

SOLUTION. First, replace each set in the expression by its R-diagram. Secondly, perform the set operation by appealing to the rule one, two and three, until a single R-diagram is obtained. Thirdly, write the set corresponding to this last diagram. Hence,

(17) $[(A \cap B)' \cap (A' \cup B)'] \cup A =$  $= A$.

It should be noted that a R-diagram with no dots occurring represents the null set, and a R-diagram with dots occurring in all portions represents the universal set.

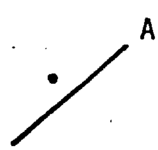
Now suppose that three sets are involved. Then U will be divided into $2^3 = 8$ disjoint parts.

S_5: 

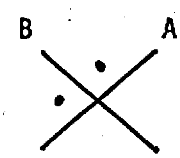
Frame 1 Frame 2 Frame 3

Referring to the sequence of frames S_5, the first frame is a VE-diagram showing the eight disjoint portions, and showing that the boundary C splits each of these four parts into two parts each. Frame 2 of S_5 shows how we may use "straight line" boundaries A and B to divide U into four parts, and how we may split each of these four parts into two portions by using four "straight segments", one segment in each of the four parts determined by A and B. Here, we note that no single "straight line" boundary can accomplish this. These four segments taken together accomplish the same objective as the single circular boundary in Frame 1. Hence, these four segments taken together form the boundary for C. We see, then, that the boundary for the set C gets "sprinkled" about. Frame 3 shows that the rectangular boundary for U has been deleted.

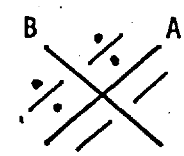
The following three diagrams illustrate what we will call: (a) primary, (b) secondary, and (c) tertiary R-diagrams, respectively.



(a)



(b)



(c)

Each of the above R-diagrams represents the same set A. This is easily seen, since in each case all portions above the line A have been dotted in. The rules one through five, above, apply to tertiary R-diagrams in exactly the same way as they do to primary and secondary R-diagrams. In fact, these five rules apply equally well to R-diagrams of all orders.

Consider the following examples relating to tertiary R-diagrams.

EXAMPLE 4. Write tertiary R-diagrams representing:
(a) $A' \cap B \cap C$, (b) $B \cap C$, and (c) $A \cup C$.

SOLUTION. (a) To write the R-diagram for $A' \cap B \cap C$, we must dot in all portions which are simultaneously below A and above B and above C. Hence,

(18) $A' \cap B \cap C =$ 

(b) To write the R-diagram for $B \cap C$, we must dot in all portions which are simultaneously above B and above C. Thus,

(19) $B \cap C =$ 

(c) To write the R-diagram for $A \cup C$, we will first dot in all portions above A, and then continue dotting until all portions above C are also dotted in. Hence,

(20) $A \cup C =$ 

EXAMPLE 5. Show that: $[(A \cap B') \cup (A' \cup B)'] \subseteq B'$.

SOLUTION. We first write the R-diagram for each side of the inclusion, and then appeal to Rule IV. Hence,

(21) $[(A \cap B') \cup (A' \cup B)'] =$  $\cup$  $=$ 

and,

(22) $B' =$ 

Comparing (21) and (22), we see that, according to Rule IV, the inclusion is true.

EXAMPLE 6. Show that: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

SOLUTION. We proceed by writing the R-diagram for each side, and then comparing them. Hence,

(23) $A \cap (B \cup C) =$  $\cap$  $=$ 

Also,

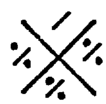
(24) $(A \cap B) \cup (A \cap C) =$  $\cup$  $=$ 

Since the diagrams in (23) and (24) are the same, we have that the equality holds, and so distributivity of intersection over union follows.

EXAMPLE 7. Write the sets indicated by the following R-diagrams.

(a)  (b) 

SOLUTION. (a) Here, we note that <u>all</u> portions <u>below A</u> have been dotted in, together with <u>all</u> portions <u>below B</u>. Hence, we have all of A' together with all of B'. So,

(25)  $= A' \cup B'$.

(b) In this case, we observe that all portions <u>above C</u> have received dots. Hence, we have all of C. Thus,

(26)  $= C.$

The preceeding examples clearly illustrate the applications and facility of R-diagrams in simplifying set-theoretic expressions and establishing set-theoretic inclusions and equalities. Some reflection and experimentation will undoubtedly guide the reader to other applications. Clearly, among these are applications to inventory problems and symbolic logic.

We close by providing the reader with a few exercises on which to practice. The reader is invited to solve the following first by R-diagrams, and then without R-diagrams.

EXERCISES. (a) Show that: $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$, where: $X \Delta Y = (X' \cap Y) \cup (X \cap Y')$ for sets X and Y. This set-theoretic operation is call symmetric difference.

(b) Show that the symmetric difference is associative.

(c) Show that: $(A \Delta B \Delta C)' = A' \Delta B \Delta C = A \Delta B' \Delta C =$

$$A \Delta B \Delta C'.$$

Reference

Randolph, John F., Cross-Examining Propositional Calculus and Set Operations, American Mathematical Monthly, Vol. 72 (1965), No. 2 (Feb), pp. 117-127.